Heat Capacity Anomalies of Superfluid ^{4}He under the Influence of a Counterflow near T_λ

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We present a thermodynamic treatment of superfluid helium in the presence of an applied heat current, Q, which produces a counterflow velocity W. Using a thermodynamic expression relating the heat capacity to the depression of ρ_s with $\ddot{W}$, we find that near T_λ, the heat capacity is expected to diverge at a depressed transition temperature. The exponent is found to be 0.5 in mean-field theory and in conventional renormalization group theory, In contrast, if $\bar{W}$ rather than Q is held constant, the heat capacity remains finite.

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Owing to the remarkable success of the renormalization group theory (R(i), phase transitions at equilibrium arc to a large extent well understood. ~'here is still much to be learned, however, concerning non-equilibrium and dynamic phenomena. Near the lambda point of ^{4}He, an applied heat flux Q can have an interesting influence on the nature of the transition. A number of experiments [1] report that the transition temperature is depressed. The depressed transition temperature $T_c(Q)$ scales with Q as $[7_a - T_c(Q)] - Q^x$. Theories [2] predict that $x = 1 / (2\nu) = 0.746$, where $\nu = 0.6705$ [3] is the correlation length exponent. Recently, 1 Iaussmann and Dohm (I ID) [4] have applied RG to this problem and predicted cusp shape curves [5] for the superfluid density and the heat capacity at various values of constant Q near $T_c(Q)$. However, it was pointed out [6] that the heat capacity anomalies predicted by IID are not consistent with the result of a simple thermodynamic calculation at constant Q. J111 responded [6] that their heat capacities were, in fact, calculated at constant superfluid velocity. 'I'his draws our attention to the fact that the basic thermodynamics of this system is not widely understood. It is the intent of this paper to fill in this gap. Surprisingly, when we. recalculate the heat capacity properly, it diverges at $T_c(Q)$ even in mean-field theory.

Liquid helium in the presence of a counterflow can be treated as a system that exhibits an extra degree of thermodynamic freedom. This is a unique case in which a dynamic situation may be treated by equilibrium thermodynamic analysis, In superfluid helium, the first law of thermodynamics at constant density may be. written as [7]:

$$dE^s = T\,dS + \vec{W} \bullet d\vec{j}_o, \tag{1}$$

where E' is the internal energy in a reference frame traveling with the superfluid, $\vec{W} = \vec{v}_n - \vec{v}_s$ is the velocity of the normal fluid in that frame, and $\vec{j}_o = \rho_n \vec{W}$ is the normal fluid momentum density. The term $\vec{W} \bullet d\vec{j}_o$ is the work per unit volume required to set the normal fluid into motion. Thus the new conjugate variables in superfluid arc $(\vec{W}, \vec{j}_o)$. Most phase transition theories, to which we wish to compare our results, assume that the normal fluid is at rest. The internal energy in the normal fluid frame E^n can be obtained using the Galilean transformation [7] $E^n = E^s + \rho \vec{W}^2 / 2 - \vec{j}_o \bullet \vec{W}$, giving

$$dE^n = TdS + \vec{P} \bullet d\vec{W} \qquad (2)$$

where $\vec{P} = \rho_s \vec{W}$. Thus in the normal fluid frame the new conjugate pair is $(\vec{P}, \vec{W})$. The free energy is $F(T, W) = En - TS$ giving:

$$dF = -SdT + \vec{P} \bullet d\vec{W}$$
$$F(T, \vec{W}) = F(T,0) + \int_0^{\vec{W}} \rho_s(\vec{W}) \vec{W} \bullet d\vec{W} \qquad (3)$$

We henceforth drop the vector notation because all motions are in the same direction in the case we treat. The term $F(T, O)$ contains all the characteristics of the phase transition at zero W, which has been well studied both experimentally and theoretically. At a finite W the only unknown is the function $\rho_s(W)$. Qualitatively, if ρ_s is only weakly depressed, the integral in cq. (3) can be approximated by $\rho_s(0)W^2 / 2$, The dashed line in Figure 1 shows $F(T, W)$ for this case. On the other hand, f ρ_s is significantly depressed (Fig. 1a), the ntegrand in eq.(3),

$\rho_s(W)W$, increases with W at small W, but might decrease at large W (Fig. lb). As shown by the solid line in Fig. 1 c, a critical counterflow velocity W_c exists when $F(T,W)$ changes from convex to concave [8]. This point is also the point where $\rho_s(W)W$ is maximum. As we shall see below, if $\rho_s(W)$ is sufficiently depressed to reach this point, a thermodynamic phase transition occurs.

The depression of ρ_s cannot be derived by thermodynamic arguments. It must be measured experiment.ally, calculated from microscopic. theory, or obtained from phase transition theory near T_λ. Experimentally, not much is known about $\rho_s(W)$. The only experimental evidence to date is the observation by Hess [9] far from T_λ, which agrees with the roton theory. Near T_λ, only theoretical predictions exist. The three existing theories are. the mean-field theory [10], the ψ theory [11], and the RG theory of HD [4]. Since. we will use the $\rho_s(W)$ expression from these theories to compute the heat capacity, it is desirable to show that the theories are consistent with thermodynamics. These theories all start from a mean-field expansion:

$$F_{mf} = \alpha|\psi|^2 - t\,\beta|\psi|^4 + \left(\hbar^2/2m\right)|\nabla\psi|^2 + M|\psi|^6. \tag{4}$$

It is not clear that F_{mf} obeys eq. (3). Here α, β and M are expansion coefficients, M is zero except in the ψ theory, the macroscopic wave function is given by $\psi = \eta e^{i\phi}$, where $\rho_s = m|\psi|^2$ and $v_s = (\hbar/m)\nabla\phi$, and m is the mass of a helium atom. In terms of ρ_s and v_s:

$$F_{mf} = \frac{\alpha\rho_s}{m} + \frac{\beta\rho_s^2}{m^2} + \frac{\rho_n v_n^2}{2} + \frac{\rho_s v_s^2}{2} + \frac{\hbar^2(\nabla\rho_s)^2}{8m^2\rho_s} + \frac{M\rho_s^3}{m^3}, \tag{s}$$

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where the term $\rho_n v_n^2 /2$ is added to account for the motion of the normal fluid. A controversy exists in the literature concerning the proper procedure for minimizing F_{mf} with respect to ψ (or ρ_s). Pitaevskii [12] minimizes F_{mf} while holding the momentum $j = P + \rho v_n$ constant. Here, F_{mf} is a free energy in the laboratory frame. Khalatnikov [13] uses a Galilean transformation to obtain a free energy in the normal fluid frame:

$$F_{mf}^n = \frac{\alpha \rho_s}{m} + \frac{\beta \rho_s^2}{m^2} + \frac{\rho_s W^2}{2} + \frac{\hbar^2 (\nabla \rho_s)^2}{8m^2 \rho_s} + \frac{M \rho_s^3}{m^3}. \tag{6}$$

He then minimizes F_{mf}^n holding W constant. To show that this is the correct approach, we note that F_{mf}^n varies with W through $\rho_s(W)$ and the term $\rho_s W^2/2$. Thus

$$\frac{dF_{mf}^n}{dW} = \frac{\partial F_{mf}^n}{\partial \rho_s}\bigg|_W \frac{d\rho_s(W)}{dW} + \frac{\partial F_{mf}^n}{\partial W}\bigg|_{\rho_s}. \tag{7}$$

From eq. (6), $\left(\partial F_{mf}^n / \partial W\right)_{\rho_s} = \rho_s(W) W$. The optimization condition is $\left(\partial F_{mf}^n / \partial \rho_s\right)_W = 0$. Thus eq. (7) and eq. (3) become the same, proving consistency with thermodynamics.

In uniform flow, $\nabla \rho_s = 0$. The expression for $\rho_s(W)$ is obtained by optimizing F_{mf}^n. All three theories give $\rho_s(W)$ of the form:

$$\rho_s(W) = \rho_s(0) f(\kappa), \tag{8}$$

where $\kappa = W/W$, and W_t is a characteristic velocity give by $W, = h / m\xi$. Below T_λ,

$\xi = \xi_o (2t)^{-\nu}$, where $\xi_o = 1.43 \times 10^{-8}$ cm [14]. The characteristic velocity W_t *can* be expressed as

$W_t = W, t'$, where $W_o = \hbar 2^\nu / m\xi_o = \textbf{175.4 mkt.}$ For the mean-field theory, $f(\kappa) = 1 - 2\kappa^*$.

For the ψ theory $f(\kappa) = -\dfrac{1-M}{2M} + \dfrac{1}{2}\sqrt{\left(\dfrac{1-M}{M}\right)^2 + \dfrac{4}{M}\left(1 - \dfrac{6+2M}{3}\kappa^2\right)}$. For HD, $f(\kappa)$ is

given by eqs **5.12,** C11 and C3 in ref. 3. All three theories predict that ρ_s is sufficiently

depressed for a phase transition to occur.

Next we compute the heat capacity using p,(W) from these theories. We first treat the

case where W is held constant. Experimentally, this might be the case of a persistent superfluid

current flowing around a loop, similar to the superfluid gyroscope experiment demonstrated by

Clew and Reppy [15]. From cq. (3) above:

$$\Delta F(T, W) = F(T, W) - F(T, 0) = \rho_s(0) W_t^2 \int_0^\kappa x f(x) dx \qquad (9)$$

The heat capacity is changed by $\Delta C_W = -TV \left.\dfrac{\partial^2 \Delta F(T, W)}{\partial T^2}\right|_W$, where $V = \textbf{27.38}$ cn$\sim^3$/mole [16] is

the molar volume. Using $p,(0) = \rho_o t^\zeta$, where $\rho_o = \textbf{0.370}$ gm/cm^3 [17], together with the

scaling relation $\zeta = \nu = (2 - et)/3$, we obtain:

$$\Delta C_W t^\alpha = -C_o \nu \left[3(3\nu - 1)\int_o^\kappa x f(x) dx - \textbf{(4}\ \nu - 1)\kappa^2 f(\kappa) + \nu\kappa^3 \frac{\partial f(\kappa)}{\partial \kappa}\right] \qquad (10)$$

where $C_o = V\rho_o W_o^2 / T_\lambda = 143$ (J / mole K). For the mean-field theory, this reduces to:

$$\Delta C_W t^\alpha = C_o \nu \kappa^2 \left[(1-\nu) + (1+\nu)\kappa^2 \right] / 2 . \tag{11}$$

For the ψ theory and for HD, equation (10) is evaluated using numerical differentiation and integration. These results arc shown in Fig. 2a. C_W approaches a finite constant at $\kappa_c = WC / W$, in all three theories.

It is experimentally feasible to measure the heat capacity in a thermal conductivity cell while passing a constant heat flux Q through it, where

$$Q = \rho_s(W) W S T , \tag{12}$$

and S $\simeq$ 1.58 J/gm K [18] is the entropy. Therefore, keeping Q constant is the same as keeping $P = \rho_s(W)W$ constant. At constant P, it is necessary to define $\Phi(T, P) = F(T, W) - WP$, giving $d\Phi = -SdT -- WdP$ and $\Delta\Phi(T, P) = \Phi(T, P) - \Phi(T, 0) = - \int_0^W W d(\rho_s W)$. Thus the heat capacity can be computed as:

$$\Delta C_Q = -TV \left[\partial^2 \Delta\Phi(T, P) / \partial T^2 \right]_Q . \tag{13}$$

Although ΔC_W is finite, ΔC_Q diverges at a critical heat current, Q_c. The reason may be seen directly from thermodynamics. Starting from the entropy S(T, *W)*, *wc* obtained the relations:

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$$\mathbf{ds} = (\partial S/\partial T)_W \, dT + (\partial S/\partial W)_T \, \mathbf{dw} \tag{14}$$

$$C_Q = T(\partial S/\partial T)_Q = C_W + T(\partial S/\partial W)_T (\partial W/\partial T)_Q \tag{15}$$

From eq. (3), $dF = -SdT + PdW$, we obtained a Maxwell relation $(\partial P/\partial T)_W = -(\partial S/\partial W)_T$
'T'bus,

$$C_Q = C_W - T(\partial P/\partial T)_W (\partial W/\partial T)_P \ . \tag{16}$$

Here we have made use of eq. (12) to obtain the relation $(\partial W/\partial T)_Q = (\partial W/\partial T)_P$. Using the chain rule:

$$(\partial P/\partial T)_W (\partial T/\partial W)_P (\partial W/\partial P)_T = -1, \tag{17}$$

$$C_Q = C_W + T(\partial P/\partial T)_W^2 / (\partial P/\partial W)_T \ . \tag{18}$$

The phase transition occurs when $(\partial^2 F/\partial W^2)_T = (\partial P/\partial W)_T = 0$. Thus C_Q diverges at this point. This result must be true for any theory that depresses ρ_s enough to reach $(\partial P/\partial W)_T = 0$, including all three theories discussed here. Equation (18) gives,

$$\Delta C_Q = \Delta C_W + C_o t^{-\alpha} v^2 \kappa_2^2 \left[\frac{\kappa \partial f(\kappa)}{\partial \kappa} - f(\kappa) \right]^2 \Big/ \frac{\partial \kappa f(\kappa)}{\partial \kappa} \tag{19}$$

'l'he results can be expressed in terms of the variable $q = Q/Q_c$ using the relation $q = \kappa f(\kappa)/[\kappa_c f(\kappa_c)]$ obtained from eq. (12). The values for $\kappa_c, f(\kappa_c)$, and Q_c/t^{2v} arc listed in 'l'able 1. For the mean-field theory:

$$t^\alpha \Delta C_Q = C_o v \kappa^* \left[\frac{(v+1) + 5(3v-1)\kappa^2 + 2(v-3)\kappa^4}{2.(1-6\kappa^2)} \right]$$
$$= (C_o/2)v(v+1)\kappa_c^2 f^2(\kappa_c)q^2 \left[1 + 0.965q^2 + ... \right],$$

(20)

at small q. Figure 2b shows that all three theories give a divergent C_Q. Again the results for the ψ theory and the HD theory arc obtained numerically. Because Q_c is different for the three theories, wc have used Q/Q_c^{HD} as the x-axis, so that all three theories can be plotted on the same scale. I lere Q_c^{HD} is the critical heat current given by HD. Near Q_c, eq. (18) gives $C_Q \sim 1/(\partial P/\partial W)_T$. We can expand P about P_c, the Superfluid momentum at the phase transition:

$$P = P + \left(\frac{\partial P}{\partial W} \right)_{W_c} (W - W_c) + \frac{1}{2} \left(\frac{\partial^2 P}{\partial W^2} \right)_{w'} (W - W_c)^2 +.$$

(21)

Since $(\partial P/\partial W)_{W_c} = 0$, and $(\partial^2 P/\partial W^2)_{W_c} < 0$, $(P_c - P) \sim (W_c - W)'$, and $(\partial P/\partial W)_T \sim (W_c - W)$. Thus:

$$C_Q \sim 1/(W_c - W) \sim 1/\sqrt{P_c - P} \sim (Q_c - Q)'' {}^u$$

(22)

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where the exponent $u = 0.5$. We have verified numerically that all three theories are consistent with this prediction. It is easy to show that if we define $\theta = [T_c(Q) - T]/T_c(Q)$, then

$$C_Q \sim \theta^{-u}. \tag{23}$$

In conclusion, our analysis has lead to a number of surprising results. There exists near T_λ, in the $T - Q$ plane, a curve $T_c(Q)$ that is apparently a new line of critical points, at which the heat capacity, C_Q, diverges according to eq.(23). Unlike other familiar phase transitions, the heat capacity divergence in this case is predicted by mean-field theory, and indeed, the exponent $u = 1/2$ can be arrived at by the mean-field type argument that leads to eq. (22). It may seem surprising that the mean-field value for u also results from the RG theory of HD[19], but perhaps that means their theory only deals with critical point behavior at the lambda-point, $t = 0$, not on the line $T_c(Q)$, where $\theta = 0$. Because of fluctuations, one dots not expect the mean-field theory exponent to be correct.

Experimental measurements of C_Q near $T_c(Q)$ are urgently needed. As our arguments have shown, they would constitute the first information concerning how ρ_s depends on W near T_λ. Existing experiments[1] show that dissipation due to vortex formation [20] tends to set in at $Q/Q_c^{HD} \sim 0.s$, except perhaps very close to T_λ, where Q_c is very small. However, Fig.2b shows that a large effect ($\Delta C_Q \sim 3$ J/mole K) may be expected even at $Q/Q_c^{HD} \sim 0.5$.

Finally, we speculate that this new phase transition involves fluctuations of a different order parameter, perhaps W or $W - WC$, with conjugate field P or P $\sim$ PC, implying a different universality class from the conventional lambda transition. Large fluctuations in W at constant

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f? (or P) are to be expected near the point $(\partial P/\partial W)_T = 0$. On the other hand, in spite of the agreement between the theories discussed here, it is also possible that $\rho_s(W)$ is never sufficiently depressed to reach the point $(\partial P/\partial W)_T = 0$, anti superflow thus breaks up in some other way.

FIGURE CAPTIONS AND TABLE

Figure 1: A plot of a) $p_s(W) / \rho_s$ (0); b) $W\rho_s(W)$ $Ip_s(0)$; and c) The free energy for the cases where: dashed line - ρ_s is not depressed; and solid line - ρ_s is depressed sufficiently for the onset of a phase transition at WC. This illustration is the result of the mean-field theory.

Figure 2: Change in the heat capacity times t^α at a) constant W, and b) constant Q. Thin line - I ID theory, thick line - mean-field theory, triangles - ψ theory with $M = 1$, dashed line - ρ_s not depressed by W as discussed in ref. [6].

Table I: A summary of κc, $f(\kappa c)$ for the three theories (M=1 for the ψ theory).

	Mean-Field	ψ Theory	IID Theory
κ_c	$1/\sqrt{6}$	0.433	**0.397**
$f(\kappa_c)$	2/3	0.707	0.790
Q_c / t^{2v} (W/cm^2)	6082	6842	7007

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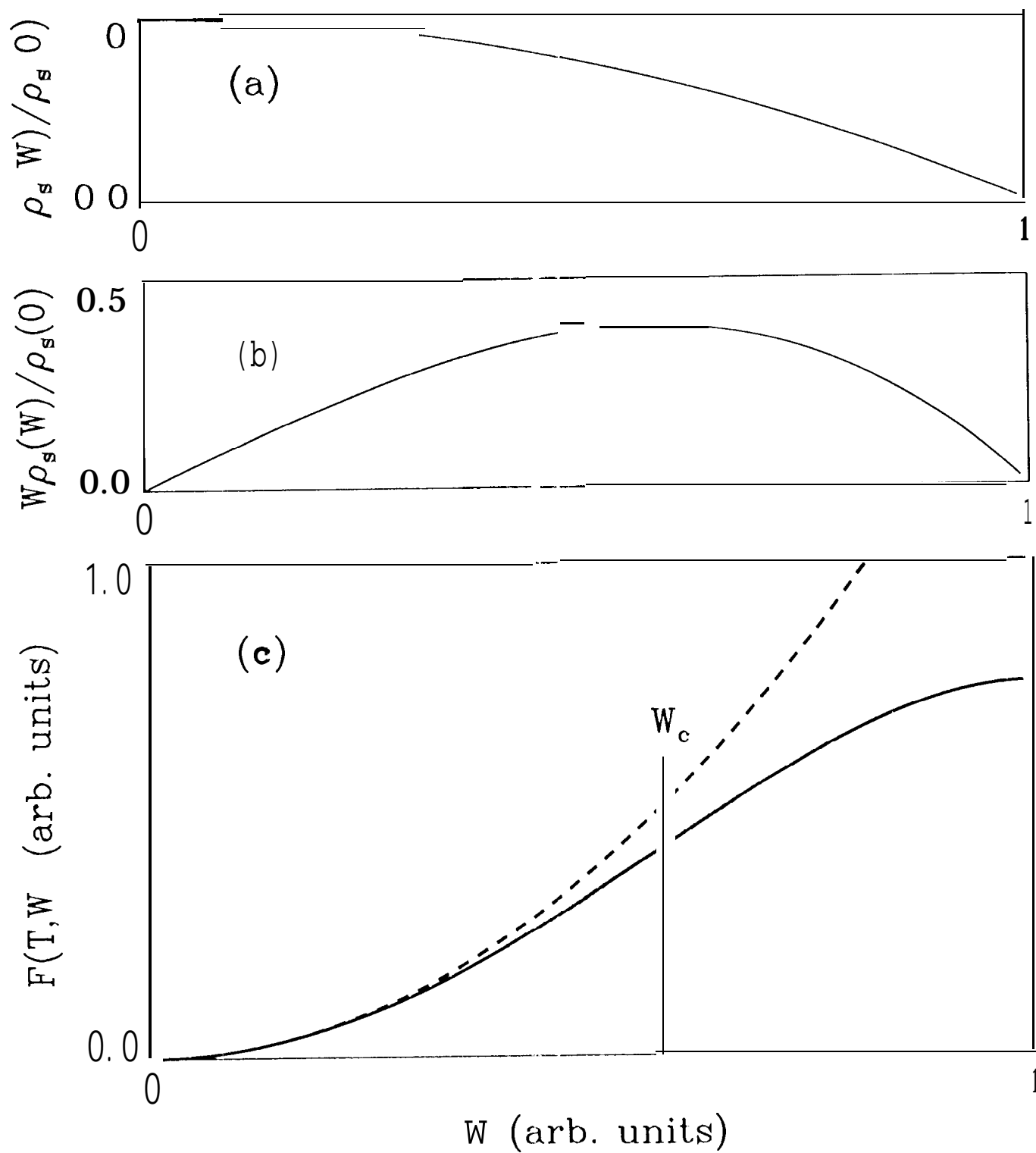20 T. C. P. Chui, U. E. Israelsson and M. J. Lysck, Physica B 194-196, 601 (1994).

Fig. 1

Fig 2

(a)

(b)